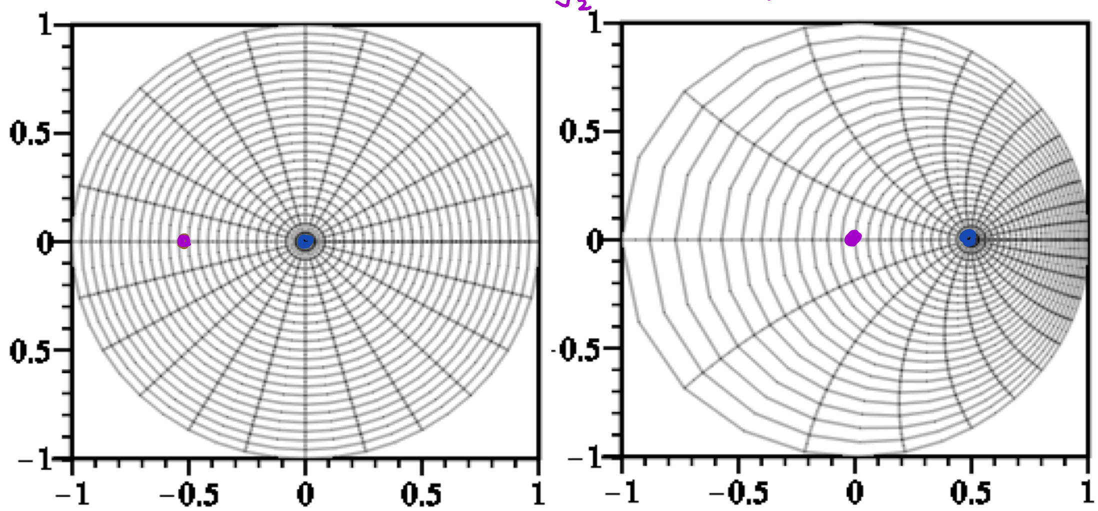
The transformation picture for $g_{\frac{1}{2}}(z)$ and $g_{-\frac{1}{2}}(z)$. Which is which?

$g_{\frac{1}{2}}(z)$



$g_{-\frac{1}{2}}(z)$

<u>Step 3</u>: Combining steps 1 and 2, we showed that for each $z_0 \in \mathrm{D}(0;1)$ *every* conformal diffeomorphism of the unit disk with

$$f(0) = z_0$$

can be written as

$$f(z) = g_{z_0}\left(\mathrm{e}^{i\theta} z\right)$$

If given such an f with $f(0) = z_0$

for some choice of θ. Proof: The composition function

$$g_{-z_0} \circ f$$

$g_{-z_0}(f(0)) = g_{-z_0}(z_0) = 0$!

is a conformal diffeomorphism of the unit disk which maps the origin to itself. Thus ✓

$$g_{-z_0} \circ f(z) = \mathrm{e}^{i\theta} z$$

$g_{z_0} \circ$

$$f(z) = g_{z_0}(\mathrm{e}^{i\theta} z).$$

3 real free parameters: $z_0 = x_0 + i y_0$ & θ

Riemann Mapping Theorem (version 1)

- Let $A \subseteq \mathbb{C}$ (but $A \neq \mathbb{C}$) be open and simply connected.
Let $z_0 \in A$. Let $\theta \in (-\pi, \pi]$.
Then $\exists!\, f: A \to D(0; 1)$ such that f is a conformal bijection satisfying

$$\bullet \quad f(z_0) = 0$$

$$\bullet \quad arg(f'(z_0)) = \theta$$

Note that this means there are three real degrees of freedom for conformal bijections with the disk: 2 from the choice of z_0 and one from the choice of the argument of $f'(z_0)$. So we probably missed some of the possibilities in our early examples. But we just proved both existence and uniqueness for conformal transformations of the unit disk though, once z_0 and θ are specified, using Mobius transformations and rotations.

hard. e.g see 6000-level course. Wikipedia

The existence part of the general proof for any open simply connected subset of $\mathbb{C}$ except $\mathbb{C}$ itself would take several lectures to explain and we won't do it in this course. But we already have the tools to prove uniqueness:

proof of uniqueness: Suppose f_1, f_2 satisfy the conditions above. Define

$$g := f_2 \circ f_1^{-1} : D(0; 1) \to D(0; 1). \quad \bullet$$

Use our earlier discussion.

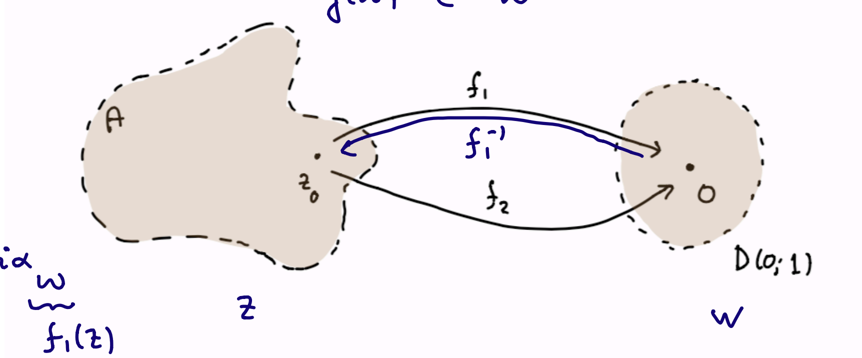
$0 \mapsto 0$

continue tomorrow.

$\Rightarrow$ ~~$g(z) = e^{i\alpha} z$~~ for some α (Monday).

$g(w) = e^{i\alpha} w$

Wed:



$$\Rightarrow f_2(\underbrace{f_1^{-1}(w)}_{:= z}) = e^{i\alpha} \underbrace{w}_{f_1(z)}$$

$$z = f_1^{-1}(w)$$

$$f_1(z) = w$$

$$\Rightarrow f_2(z) = e^{i\alpha} f_1(z) \quad \forall z \in A$$

$$\Rightarrow f_2'(z_0) = e^{i\alpha} f_1'(z_0)$$

$$\underbrace{\arg f_2'(z_0)}_{\theta} = \alpha + \underbrace{\arg f_1'(z_0)}_{\theta} \Rightarrow \alpha = 0 \qquad \text{so } f_2 = f_1$$

Riemann Mapping Theorem (version 2)

Let $A, B \subseteq \mathbb{C}$ be open and simply connected but not all of $\mathbb{C}$.

Let $z_0 \in A$, $w_0 \in B$, $\theta \in (-\pi, \pi]$.

Then $\exists!\, f: A \to B$ such that f is a conformal bijection satisfying

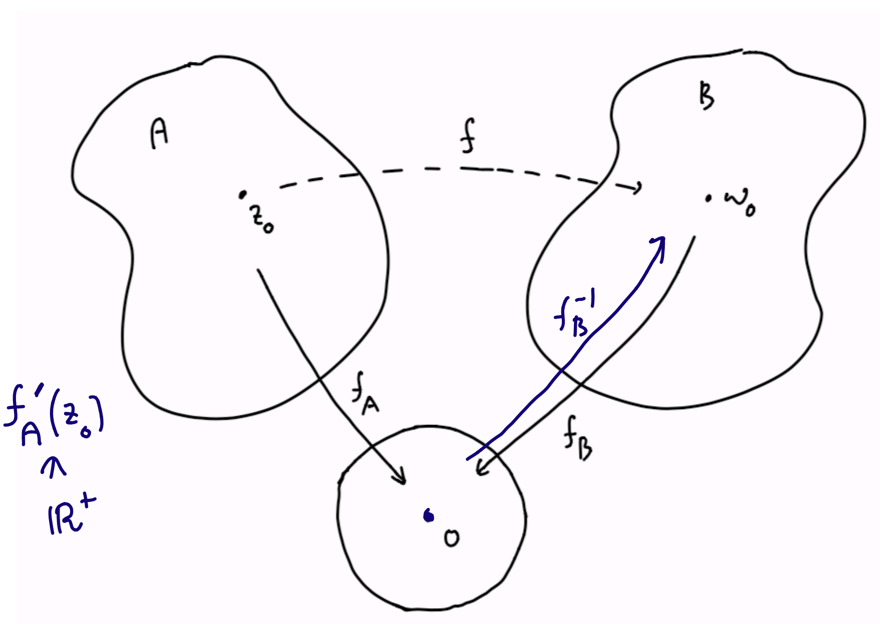
$$f(z_0) = w_0$$
$$arg(f'(z_0)) = \theta$$

proof: Chase the diagram arrows below to prove existence and uniqueness from version1 of the RMT: Letting f_A, f_B be as in version 1 on the previous page, say with arguments of the derivative at z_0 both equal to zero; along with a rotation of the unit disk. In other words, show that

$$* \quad f(z) = f_B^{-1}\left(e^{i\theta} f_A(z)\right)$$

is the unique conformal transformation that works.

$f_A'(z_0) \in \mathbb{R}^+ \quad (\theta = 0)$
$f_B'(w_0) \in \mathbb{R}^+$



Let's check that specified f works.

$z_0 \mapsto w_0$ ✓

$$f'(z_0) = \underbrace{(f_B^{-1})'(0)}_{\frac{1}{f_B'(z_0)} \,\in\, \mathbb{R}^+} \cdot e^{i\theta} \underbrace{f_A'(z_0)}_{\in\, \mathbb{R}^+}$$

$\Rightarrow \arg f'(z_0) = \theta$ ✓

skip: this is the only possible f.

filled in after class. If $\exists$ f conformal equivalence $f: A \to O$, $f(z_0) = w_0$, $\arg f'(z_0) = \theta$

then consider

$$f_B \circ f \circ f_A^{-1} : D(0;1) \to D(0;1)$$
$$0 \mapsto 0$$

$$\Rightarrow f_B(f(\underbrace{f_A^{-1}(w)}_{:= z})) = e^{i\alpha} \underbrace{w}_{= f_A(z)} \qquad \text{Schwarz' Lemma, some } \alpha$$

$$f_B(f(z)) = e^{i\alpha} f_A(z)$$

f_B^{-1}: $\Rightarrow f(z) = f_B^{-1}(e^{i\alpha} f_A(z))$, and by repeating $\arg f'(z_0) = \theta$ discussion, $\alpha = \theta$.

It turns out that the maps we were missing in the examples from the start of class were compositions of the ones we found, with *fractional linear transformations*, of which Mobius transformations are examples.

Def a *fractional linear transformation (FLT)* $f: \mathbb{C} \cap \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function defined by

- $$f(z) = \frac{a\,z + b}{c\,z + d},$$

e.g. Möbius $g_{z_0}(z) = \dfrac{z + z_0}{1 + \bar{z}_0 z}$

where a, b, c, $d \in \mathbb{C}$ and

- $$ad - bc = det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0 .$$

Note that when the determinant does equal zero, the function f is just a constant. Also, one could normalize the determinant to be 1 by dividing all of the coefficients by the same number (a square root of the determinant).

- Example $f(z) = a\,z + b = \dfrac{a\,z + b}{0\,z + 1}$. You will show in your homework that these are
- the only one-to-one conformal maps defined on all of $\mathbb{C}$. Notice that they are conformal bijections of $\mathbb{C}$.

$f' \neq 0$

- Exercise Why is there no conformal bijection $f: \mathbb{C} \to D(0;1)$? It's a one-line answer if you can think of it.

If f existed it would be a bounded entire fcn
those are constants !!
(Liouville)

The algebra of fractional linear transformations: Show that if

$$f(z) = \frac{a\,z + b}{c\,z + d}$$

$$g(w) = \frac{\alpha\, w + \beta}{\gamma\, w + \delta}$$

Then

$$g(f(z)) = \frac{A\,z + B}{C\,z + D} = g\left(\frac{az+b}{cz+d}\right) = \frac{\alpha\left(\frac{az+b}{cz+d}\right) + \beta}{\gamma\left(\frac{az+b}{cz+d}\right) + \delta} \cdot \frac{cz+d}{cz+d}$$

where

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \underset{\downarrow g}{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}} \underset{\downarrow f}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}.$$

$$= \frac{\alpha(az+b) + \beta(cz+d)}{\gamma(az+b) + \delta(cz+d)} = \frac{(\alpha a + \beta c)z + (\alpha b + \beta d)}{(\gamma a + \delta c)z + (\gamma b + \delta d)}$$

A matrix for the composition FLT can be obtained by multiplying the matrices for the individual FLTs!

Geometers would say: The *group* $SL(2, \mathbb{C})$ ("The *special linear group* of 2×2 matrices with entries from $\mathbb{C}$) *acts* on the Riemann sphere $\mathbb{C} \cup \{\infty\}$. The word *special* refers to the fact that each FLT can be represented almost uniquely ~~uniquely~~ with a matrix having determinant exactly equal to 1, and the word *acts* refers to the fact that matrix multiplication (the group operation) in the group, corresponds to composition of transformations on the Riemann sphere.

To be continued .... this algebra makes composing and finding inverses for FLT's straightforward. On Wednesday we'll continue this discussion and explain interesting geometry related to FLT's.

Math 4200
Wednesday November 25
Chapter 5: 5.1-5.2 conformal maps and fractional linear transformations, continued.
We'll begin by finishing the discussion in Monday's notes. Then we'll discuss the algebra and geometry of linear fractional transformations, which are introduced there.
generalize Möbius transformations.

Announcements:
- quiz today on §5.1-5.2 material ~ canceled
- §4.4 hw due Fri @ 11:59 pm.
- §5.1-5.2 hw due next Fri
- Jess, Charlie, Austin: Wed next week zeta fcn & prime # thm
- Monday: half material from today's notes, half Daniel "theta fcns".

Math 4200-001
Week 14 concepts and homework
5.1-5.2
Due Thursday December 3 at 11:59 p.m.

5.1: 10, 11, 12.
5.2 1, 4a, 6, 7, 9, 10, 17, 24, 26, 33, 34

Continuing the discussion from Monday,

Example Let

$$f(z) = \frac{a\,z + b}{c\,z + d}$$

$$\mathrm{Id}(z) = \frac{z + 0}{0z + 1}$$

be a (non-constant) FLT. Use matrix algebra to find a formula for $f^{-1}(z)$.

matrix for f

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

a matrix for f^{-1}

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} *$$

$$f^{-1}(z) = \frac{dz - b}{-cz + a}.$$

e.g.

$$g_{\frac{1}{2}}(z) = \frac{z + \frac{1}{2}}{1 + \frac{1}{2}z} \qquad \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$g_{-\frac{1}{2}}(z) = \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z} \qquad \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} *$$

<u>Corollary</u> Fractional linear transformations are bijections of the Riemann sphere $\mathbb{C} \cup \{\infty\}$. In fact, regarding the Riemann sphere as a *Riemann surface* (see later discussion), it turns out that these ~~all of~~ are the only conformal bijections of the Riemann sphere with itself.

no quiz today :(

<u>Theorem</u> Fractional linear transformations map the set of all circles and lines to itself.

proof: Any circle or line in the $x-y$ plane can be described implicitly as the solution set to an equation

$$(1) \qquad A\left(x^2+y^2\right)+B\,x+C\,y+\mathrm{D}=0$$

where A, B, C, $\mathrm{D} \in \mathbb{R}$ and are not all zero.

We already know that translating or rotating circles (resp. lines) yields circles (resp. lines). So the Theorem holds for the first two transformations below. Show it also holds for inversions, the third transformation.

- $T_1(z)=z+a$ (translation) circles → circles, lines → lines
- $T_2(z)=c\,z$ (rotation-dilation) " "
- check! $T_3(z)=\dfrac{1}{z}$ (inversion)

convert the solution set of an equation of form (1) into the solutions set of a (different) equation of form (1).

$$T_3(x+iy)=\frac{1}{x+iy}\,\frac{x-iy}{x-iy}=\frac{x-iy}{x^2+y^2}=\frac{x}{x^2+y^2}+i\left(\frac{-y}{x^2+y^2}\right)$$

$$=u+iv$$

$$u^2+v^2=\frac{1}{x^2+y^2}$$

soln set

$\div\ x^2+y^2$

$$A(x^2+y^2)+Bx+Cy+D=0$$

$$A+B\frac{x}{x^2+y^2}+C\frac{y}{x^2+y^2}+\frac{D}{x^2+y^2}=0$$

$$A+Bu-Cv+D(u^2+v^2)=0$$

$\blacksquare$

Then show that any fractional linear transformation

$$f(z)=\frac{a\,z+b}{c\,z+d}=\frac{a}{c}+\left(b-\frac{ad}{c}\right)\frac{1}{cz+d} \quad *$$

is a composition of translations, rotation-dilations, and inversions. Hint: Treat $c=0$, $c\neq 0$ separately. If $c\neq 0$ first do something equivalent to long division to rewrite f.

$$\begin{array}{r|l} & a/c \\ cz+d & az+b \\ & az+ad/c \\ \hline & b-ad/c \end{array}$$

$f_1(z_1)=c\,z_1=z_2$

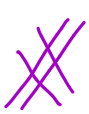
$f_2(z_2)=z_2+d=z_3$

$f_3(z_3)=\frac{1}{z_3}=z_4$

$f_4(z_4)=\left(b-\frac{ad}{c}\right)z_4=z_5$

$f_5(z_5)=z_5+\frac{a}{c}$

$f=f_5\circ f_4\circ f_3\circ f_2\circ f_1$!



to be continued.

Notice that

$$f(z)=\frac{z-a}{z-b}\left(\frac{c-b}{c-a}\right)$$

maps

$$a\to 0$$
$$b\to\infty$$
$$c\to 1.$$

Since 3 points uniquely determine particular circles one can use FLT's to map any circle or line to any other circle or line.

Using functions of this form, and their inverses, one can construct FLT's to map triples of points to triples of points:

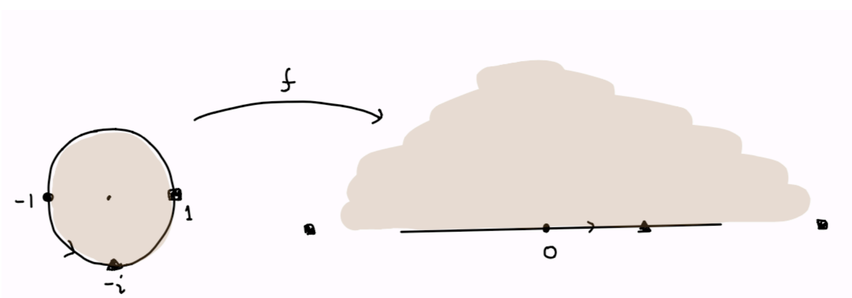
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}\to\begin{bmatrix} d \\ e \\ f \end{bmatrix}.$$

Thus you can map any line or circle to any other line or circle.

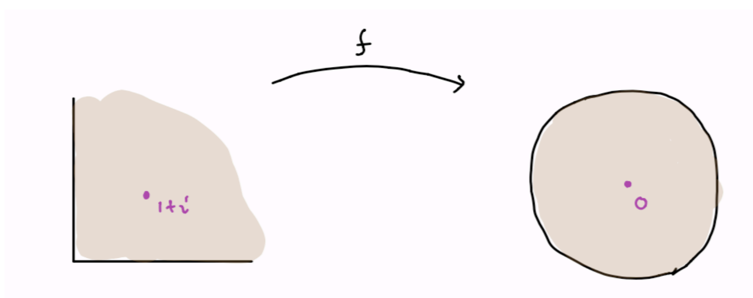
Example Find a FLT from the unit disk to the upper half plane by mapping

$$-1\to 0$$
$$1\to\infty$$
$$-i\to 1$$

and making any necessary adjustments. (By magic, once you know the boundary of the disk goes to the real axis, you only have to check that one interior point goes to an interior point, or that the orientation is correct along the boundary, to know that you're mapping the unit disk to the upper half plane instead of the lower half plane. The proof of the magic theorem is an appendix in today's notes.)



Example Find a conformal transformation of the first quadrant to the unit disk, so that the image of $1 + i$ is the origin. How many such conformal transformations are there? It's fine to write your transformation as a composition.



Riemann surfaces: These are special cases of *two-dimensional differentiable manifolds,* in the case that the *transition functions* between atlas pages are all conformal diffeomorphisms. (See Wikipedia.)

Definition A *Riemann surface* S is a topological space S together with an *atlas* consisting of *charts* $\{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$ where

(1) $\bigcup_{\alpha \in A} U_\alpha = S$ and each U_α is open.

(2) Each $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{C}$ is a *homeomorphism.* We can call the sets V_α *pages* of the atlas.

(3) The *transition maps* between parts of the pages of the *atlas* $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are all conformal.

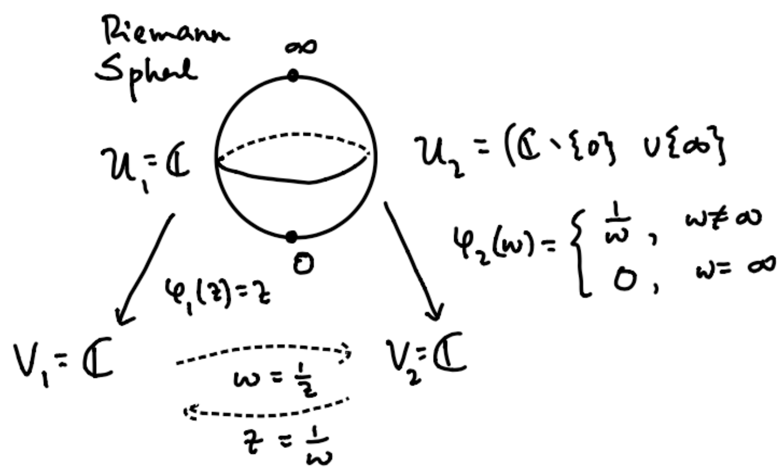
This definition makes sense when you think of what an actual geographical atlas is, along with a few concrete examples including the Riemann sphere:

- The complex plane itself, or any open set in the complex plane is a Riemann surface which has one possible atlas consisting of a single page, with $U = V$ and $\varphi = id$.

- The Riemann sphere $\mathbb{C} \cup \{\infty\}$, which is homeomorphic to the unit sphere in $\mathbb{R}^3$, as we've discussed. The easiest atlas to use has two pages:

$$U_1 = \mathbb{C},\ \varphi_1 : U_1 \to V_1 = \mathbb{C},$$
$$\varphi_1(z) = z$$
$$U_2 = (\mathbb{C} \backslash \{0\}) \cup \{\infty\},\ \varphi_2 : U_2 \to V_2 = \mathbb{C}$$
$$\varphi_2(z) = \begin{cases} \dfrac{1}{z} & z \neq \infty \\ 0 & z = \infty \end{cases}$$

Then $U_1 \cap U_2$ is the punctured complex plane $\mathbb{C} \setminus \{0\}$ and $\varphi_2 \circ \varphi_1^{-1}(z) = \dfrac{1}{z}$; $\varphi_1 \circ \varphi_2^{-1}(z) = \dfrac{1}{z}$.



Definition: Let S_1, S_2 be Riemann surfaces, and $f: S_1 \to S_2$ be a function. Then f is *analytic* if and only if each of the corresponding maps from atlas pages of S_1 to atlas pages of S_2 are analytic. Precisely, given an atlas for S_1:

$$\left\{U_\alpha,\ \varphi_\alpha : U_\alpha \to V_\alpha\right\}_{\alpha \in A}$$

and and atlas for S_2

$$\left\{O_\beta,\ \chi_\beta : O_\beta \to W_\beta\right\}_{\beta \in B}$$

then f is defined to be analytic if and only if each triple composition

$$\chi_\beta \circ f \circ \varphi_\alpha^{-1} : V_\alpha \to W_\beta$$

is analytic.

$$\begin{tikzcd}
S_1 \arrow[rr, "f"] & & S_2 \\
U_\alpha \arrow[u, phantom, "\subset" sloped] & & O_\beta \arrow[u, phantom, "\subset" sloped] \arrow[d, "\chi_\beta"] \\
V_\alpha \subset \mathbb{C} \arrow[u, "\varphi_\alpha^{-1}"] \arrow[rr, dashed, "\chi_\beta \circ f \circ \varphi_\alpha^{-1}"'] & & W_\beta \subset \mathbb{C}
\end{tikzcd}$$

So for a function $f: \mathbb{C} \to \mathbb{C} \cup \infty$ there are two cases to consider, in order to deduce whether f is analytic near z_0, as a map of Riemann surfaces:

$f(z_0) \in \mathbb{C}$: usual definition.

$f(z_0) \notin \mathbb{C}$ or undefined: Does $\dfrac{1}{f(z)}$ have a removable singularity at z_0? In other words does $f(z)$ have a pole at z_0, so that $f(z_0) = \infty$?

The text defined a *meromorphic function* on $\mathbb{C}$ to be one which is analytic except for a countable number of pole singularities. This corresponds to $f: \mathbb{C} \to \mathbb{C} \cup \infty$ being analytic as a function between Riemann surfaces.

For a function $f:\mathbb{C}\cup\infty\rightarrow\mathbb{C}\cup\infty$ there are two additional cases to consider to decide whether f is analytic as a function between Riemann surfaces:

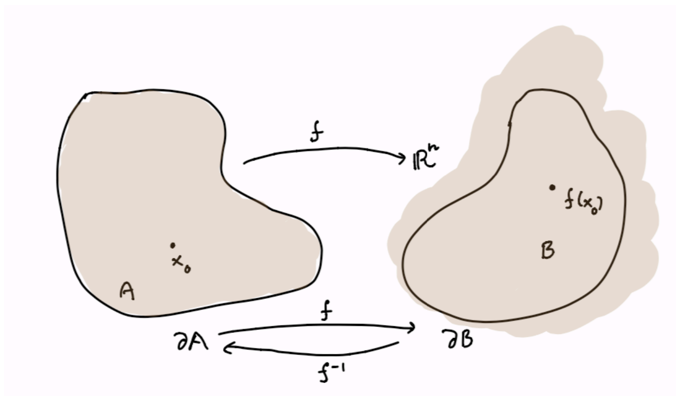
$z_0=\infty, f(z_0)\in\mathbb{C}$: Does $f\left(\frac{1}{z}\right)$ have a removable singularity at $z=0$?

$z_0=\infty, f(z_0)\notin\mathbb{C}$: Does $\dfrac{1}{f\left(\frac{1}{z}\right)}$ have a removable singularity at $z=0$?

Appendix: Magic Theorem Let $A, B \subseteq \mathbb{R}^n$ be open, connected, bounded sets.
Let $f: A \to \mathbb{R}^n$, $f \in C^1$, with $df_x : T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R}^n$ invertible $\forall\, x \in A$ (i.e. the Jacobian matrix is invertible). Furthermore, assume

- $f: \overline{A} \to \mathbb{R}^n$ is continuous and one-to-one.
- $f(\delta A) = \delta B$
- $f(x_0) \in B$ for at least one $x_0 \in A$.

Then $f(A) = B$ and f is a global *diffeomorphism* between A and B. (i.e. $f^{-1} : B \to A$ is also differentiable), and $f^{-1} : \overline{B} \to \overline{A}$ is continuous.



proof: Step 1: $f(A) \subseteq B$.
proof: Let
$$\mathsf{O} := \{x \in A \mid f(x) \in B\}$$
Then

- $x_0 \in \mathsf{O}$
- O is open by the local inverse function theorem, since $x_1 \in \mathsf{O}$ and $f(x_1) \in B$ implies there is a local inverse function from an open neighborhood of $f(x_1)$ in B, back to a neighborhood of x_1 in A.
- O is closed in A because if $\{x_k\} \subseteq \mathsf{O}$, $\{x_k\} \to x \in A$ then $\{f(x_k)\} \to f(x)$ and since $\{f(x_k)\} \subseteq B$ we have $f(x) \in \overline{B}$. But since f is one-one and maps the boundary of A bijectively to the boundary of B, $f(x)$ cannot be in the boundary of B. Thus $f(x) \in B$.
- Thus, since A is connected, O is all of A, and $f(A) \subseteq B$.

Step 2: $f(A) = B$.
proof:

- $f(A)$ is open (by the local inverse function theorem again), so $f(A) \subseteq B$ is open.
- And $f(A)$ is closed in B because if
$$\{f(x_k)\} = \{y_k\} \subseteq f(A), \text{ with } \{y_k\} \to y \in B,$$
then because $\overline{A}$ is compact, a subsequence $\{x_{k_j}\} \to x \in \overline{A}$ with $\{f(x_{k_j})\} \to f(x) = y$, so $x \notin \delta A$ because $y \in B$, so $x \in A$ and $y \in f(A)$.
- So, because B is connected, $f(A)$ is all of B.

QED.

Remark: In $\mathbb{C}$ you can also imply this theorem to unbounded domains, i.e. in $\mathbb{C} \cup \{\infty\}$ because of the following diagram, in which $f_2 \circ f \circ f_1^{-1}$ satisfies the hypotheses of the original theorem: